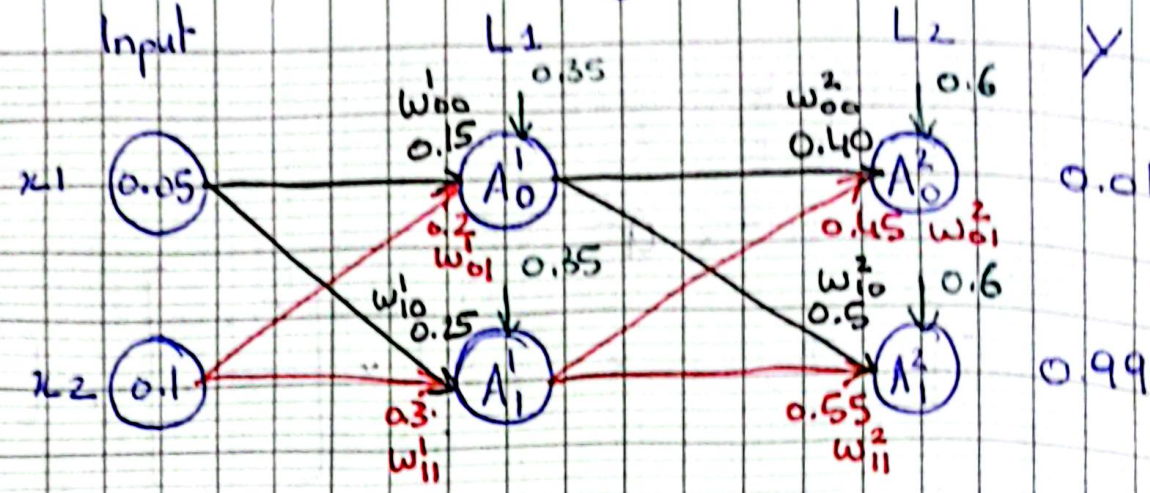
FeedForward + BackPropagation Exercise



→ Representations

$$X = \begin{bmatrix} 0.05 \\ 0.1 \end{bmatrix} \quad W^1 = \begin{bmatrix} 0.15 & 0.2 \\ 0.25 & 0.3 \end{bmatrix} \quad b^1 = \begin{bmatrix} 0.35 \\ 0.35 \end{bmatrix}$$

$$Y = \begin{bmatrix} 0.01 \\ 0.99 \end{bmatrix} \quad W^2 = \begin{bmatrix} 0.40 & 0.45 \\ 0.5 & 0.55 \end{bmatrix} \quad b^2 = \begin{bmatrix} 0.6 \\ 0.6 \end{bmatrix}$$

$\alpha = 0.5$

→ Forward Pass

↳ L1 (H.L)

$$z^1 = w^1 \times X + b^1$$

$$z^1 = \begin{bmatrix} 0.15 & 0.2 \\ 0.25 & 0.3 \end{bmatrix} \times \begin{bmatrix} 0.05 \\ 0.1 \end{bmatrix} + \begin{bmatrix} 0.35 \\ 0.35 \end{bmatrix}$$

$$z^1 = \begin{bmatrix} 0.15(0.05) + 0.2(0.1) \\ 0.25(0.05) + 0.3(0.1) \end{bmatrix} + \begin{bmatrix} 0.35 \\ 0.35 \end{bmatrix}$$

$$z^1 = \begin{bmatrix} 0.0275 \\ 0.0425 \end{bmatrix} + \begin{bmatrix} 0.35 \\ 0.35 \end{bmatrix} = \begin{bmatrix} 0.3775 \\ 0.3925 \end{bmatrix}$$

$$A^1 = \sigma(Z^1) = 1/1+e^{-Z^1}, \quad A^1 = \begin{bmatrix} 0.593 \\ 0.596 \end{bmatrix}$$

↳ L2 (Output L)

$$Z^2 = W^2 \times A^1 + b^2$$

$$Z^2 = \begin{bmatrix} 0.40 & 0.45 \\ 0.5 & 0.55 \end{bmatrix} \times \begin{bmatrix} 0.593 \\ 0.596 \end{bmatrix} + \begin{bmatrix} 0.6 \\ 0.6 \end{bmatrix}$$

$$Z^2 = \begin{bmatrix} 0.40(0.593) + 0.45(0.596) \\ 0.5\ (0.593) + 0.55(0.596) \end{bmatrix} + \begin{bmatrix} 0.6 \\ 0.6 \end{bmatrix}$$

$$Z^2 = \begin{bmatrix} 1.1054 \\ 1.2243 \end{bmatrix}$$

$$A^2 = \sigma(Z^2) = \begin{bmatrix} 0.7513 \\ 0.7728 \end{bmatrix}$$

→ Cost Calculation: $\frac{1}{2}(Y - \hat{Y})^2$ "MSE"

Error $A_0^2 = \frac{1}{2}(0.01 - 0.7513)^2 = 0.2748$

Error $A_1^2 = \frac{1}{2}(0.99 - 0.7728)^2 = 0.0236$

$$J = EA_0^2 + EA_1^2 = 0.2984$$

→ Backward Pass

1) Output Layer

* consider w_{00}^2

$$\frac{dJ}{dw_{00}^2} = \frac{dJ}{dA_0^2} \cdot \frac{dA_0^2}{dZ_0^2} \cdot \frac{dZ_0^2}{dw_{00}^2}$$ Chain Rule

$$J = \frac{1}{2}(Y_0 - A_0^2)^2 + \frac{1}{2}(Y_1 - A_1^2)^2$$

1st term → $\frac{\partial J}{\partial A_0^2} = \frac{\partial}{\partial A_0^2}\frac{1}{2}(Y_0 - A_0^2)^2 + \frac{\partial}{\partial A_0^2}\frac{1}{2}(Y_1 - A_1^2)^2$ (the second term → 0)

$$\frac{\partial J}{\partial A_0^2} = \frac{\partial}{\partial A_0^2}\frac{1}{2}\underbrace{(Y_0 - A_0^2)}_{u}{}^2$$

↳ $(\frac{1}{2}u^2)' = \frac{2}{2}\cdot u \cdot u'$

$$\frac{\partial J}{\partial A_0^2} = \frac{\partial}{A_0^2}\ \frac{2}{2}\cdot(Y_0 - A_0^2)\cdot(-1)$$

$$\frac{\partial J}{\partial A_0^2} = A_0^2 - Y_0 = 0.7513 - 0.01 = 0.7413$$

2nd term → $\frac{\partial A_0^2}{\partial Z_0^2}$, $A_0^2 = 1/1+e^{-Z_0^2}$

$$\frac{\partial A_0^2}{\partial Z_0^2} = A_0^2(1 - A_0^2) = 0.7513(1 - 0.7513)$$
$$= 0.1868$$

3rd term → $\frac{\partial Z_0^2}{\partial w_{00}^2}$, $Z_0^2 = w_{00}^2 \times A_0^1 + w_{01}^2 \times A_1^1 + b_0^2$ (the $w_{01}^2 \times A_1^1$ and b_0^2 terms → 0)

$$\frac{\partial Z_0^2}{\partial w_{00}^2} = A_0^1 = 0.593$$

⇒ All Together

$$\frac{\partial J}{\partial w_{00}^2} = 0.7413 \times 0.1868 \times 0.593 = \boxed{0.0821}$$

* Consider w_{01}^2

$$\partial J/\partial w_{01}^2 = \partial J/\partial A_0^2 \cdot \frac{\partial A_0^2}{\partial Z_0^2} \cdot \frac{\partial Z_0^2}{\partial w_{01}^2}$$

$$= 0.7413 \times 0.1868 \times 0.596 = \boxed{0.0225}$$ (A_1^1)

* Consider w_{10}^2

$$\frac{\partial J}{\partial w_{10}^2} = \frac{\partial J}{\partial A_1^2} \cdot \frac{\partial A_1^2}{\partial Z_1^2} \cdot \frac{\partial Z_1^2}{\partial w_{10}^2}$$

$A_1^2 - Y_1 = 0.7728 - 0.99 = -0.2172$

$A_1^2(1 - A_1^2) = 0.7728(1 - 0.7728) = 0.1756$

$A_0^1 = 0.593$

$$\rightarrow \frac{\partial J}{\partial w_{10}^2} = -0.2172 \times 0.1756 \times 0.593 = -0.0226$$

* Consider w_{11}^2

$$\frac{\partial J}{\partial w_{10}^2} = \frac{\partial J}{\partial A_1^2} \cdot \frac{\partial A_1^2}{\partial Z_1^2} \cdot \frac{\partial Z_1^2}{\partial w_{11}^2} = -0.2172 \times 0.1756 \times 0.596$$ (A_1^1)

$$= -0.0227$$

Updates

$w_{00}^2 := w_{00}^2 - \alpha \, \partial J/\partial w_{00}^2 = 0.4 - 0.5 \times 0.0821 = 0.3589$

$w_{01}^2 := w_{01}^2 - \alpha \, \partial J/\partial w_{01}^2 = 0.45 - 0.5 \times 0.0825 = 0.4087$

$w_{10}^2 := w_{10}^2 - \alpha \, \partial J/\partial w_{10}^2 = 0.5 - 0.5 \times (-0.0226)$

$= 0.5113$

$w_{11}^2 := w_{11}^2 - \alpha \, \partial J/\partial w_{11}^2 = 0.55 - 0.5 \times (-0.0227)$

$= 0.5613$

2) Hidden Layer

- Consider w^1_{00}

$$\frac{\partial J}{\partial w^1_{00}} = \frac{\partial J}{\partial A^1_0} \cdot \frac{\partial A^1_0}{\partial Z^1_0} \cdot \frac{\partial Z^1_0}{\partial w^1_{00}}$$

1st term →
$$\frac{\partial J}{A^1_0} = \frac{\partial J}{\partial Z^2_0} \cdot \frac{\partial Z^2_0}{\partial A^1_0} + \frac{\partial J}{\partial Z^2_1} \cdot \frac{\partial Z^2_1}{\partial A^1_0}$$

$$\partial J / \partial Z^2_0 = \frac{\partial J}{\partial A^2_0} \times \frac{\partial A^2_0}{\partial Z^2_0} = 0.7413 \times 0.1868 = 0.1384$$

$$Z^2_0 = w^2_{00} \cdot A^1_0 + w^2_{01} \cdot A^1_1 + b^2_0$$

$$\frac{\partial Z^2_0}{\partial A^1_0} = w^2_{00} = 0.4$$

$$\frac{\partial J}{\partial Z^2_1} = \frac{\partial J}{\partial A^2_1} \times \frac{\partial A^2_1}{\partial Z^2_1} = -0.2172 \times 0.1756 = -0.0381$$

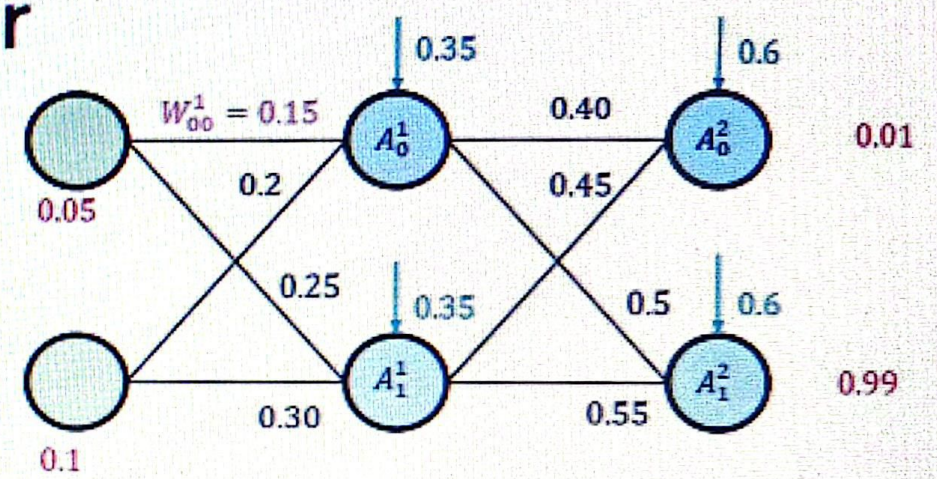
$$\frac{\partial Z^2_1}{\partial A^1_0} = 0.5$$

$$\Rightarrow \frac{\partial J}{A^1_0} = (0.1384)(0.4) + (-0.0381)(0.5) = 0.0363$$

Backward Pass - Hidden layer

$$\frac{\partial J}{\partial W_{00}^1} = \frac{\partial J}{\partial A_0^1} * \frac{\partial A_0^1}{\partial Z_0^1} * \frac{\partial Z_0^1}{\partial W_{00}^1}$$

$$A_0^1 = \frac{1}{1 + e^{-Z_0^1}}$$



2nd term ->

$$\frac{\partial A_0^1}{\partial Z_0^1} = A_0^1(1 - A_0^1) = 0.5932699(1 - 0.59326999) = 0.241300709$$

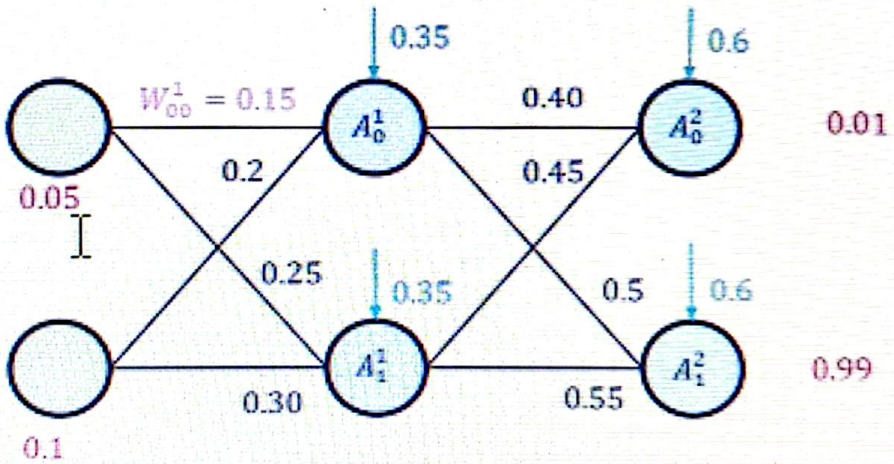
$$Z_0^1 = W_{00}^1 \times x_0 + W_{01}^1 \times x_1 + b_0^1$$

$$\frac{\partial Z_0^1}{\partial W_{00}^1} = x_0 = 0.05$$ <- 3rd term

Backward Pass - Hidden layer

$$\frac{\partial J}{\partial W_{00}^1} = \frac{\partial J}{\partial A_0^1} * \frac{\partial A_0^1}{\partial Z_0^1} * \frac{\partial Z_0^1}{\partial W_{00}^1}$$

$$= 0.036350306 * 0.241300709 * 0.05 = 0.000438568$$



Weight Update

$$W_{00}^{2} := W_{00}^{2} - \alpha \frac{\partial J}{\partial W_{00}} = 0.4 - 0.5 * 0.082167041 = 0.35891648$$

$$W_{00}^{1} := W_{00}^{1} - \alpha \frac{\partial J}{\partial W_{00}^{1}} = 0.15 - 0.5 * 0.000438568 = 0.149780716$$

$$W_{00}^{2} = 0.35891648 \qquad W_{00}^{1} = 0.149780716$$

$$W_{01}^{2} = 0.408666186 \qquad W_{01}^{1} = 0.19956143$$

$$W_{10}^{2} = 0.511301270 \qquad W_{10}^{1} = 0.24975114$$

$$W_{11}^{2} = 0.561370121 \qquad W_{11}^{1} = 0.29950229$$